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ON MATCHED ASYMPTOTIC EXPANSIONS AND THE CALCULUS OF VARIATIONS

by

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and
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Die Luft der Freiheit weht

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ABSTRACT

It is shown, with the aid of two examples, that the calculus of variations may be used to an advantage over the method of matched asymptotic expansions in solving singular boundary-layer problems of applied mechanics.

INTRODUCTION

There are only few methods available to solve singular boundary-layer problems in applied mechanics [1] to [2]. One of them is the method of matched asymptotic expansions.

The problem usually consists in finding a solution to certain differential equations (subject to appropriate boundary conditions), in which the highest derivative is multiplied by a small parameter. One then tries to use a regular perturbation scheme to solve the problem and finds several solutions, none of them being valid over the entire region of interest. However, these solutions usually contain some unknown constants which are determined by matching those solutions with each other over their common domain of validity.

In what follows, we will show with the aid of two boundary value problems, that, instead of using the matched asymptotic expansion method to find those unknown constants, the calculus of variation may be used. This means that the calculating procedure is thus reduced to a systematic series of steps not requiring any particular skill or experience. Of course, the procedure to be described may be used only in those problems where a variational formulation may be made effected, such as in potential flow problems in fluid dynamics and in conservative problems of solid-body mechanics.

The two examples considered here are: 1) Cylindrical deflection of a slightly stiff membrane, and 2) Torsion of a shaft with a keyway. The first problem is described by an ordinary differential equation, whereas the second one is two-dimensional and deals with a partial differential equation. Regular perturbations are used, in both cases, to find solutions

far and near singularities and the Rayleigh-Ritz method is used then to match the two solutions and find the unknown constants they contain. Even though both problems are linear, the procedure is by no means restricted to problems of this class.

1. DEFLECTION OF A SLIGHTLY STIFF MEMBRANE

Suppose that a pressurized membrane (Fig. 1), instead of being perfectly flexible, has a slight bending stiffness. Without loss of generality the governing equation for the deflection w may be assumed to have the following form [4]:

$$\varepsilon \frac{d^4 w}{dx^4} - \frac{d^2 w}{dx^2} = 1 \tag{1}$$

The boundary conditions for clamped edges are

$$w = \frac{dw}{dx} = 0 \qquad \text{at} \quad x = \pm 1 \tag{2}$$

The first term in eq. (1) represents the effect of bending rigidity (ε being a small parameter, $<< 1$). The second term describes the effect of membrane forces, and the third term the effect of transverse loading, which is taken to be uniform for simplicity.

If one tries to solve the problem by using a regular perturbation scheme, such as

$$w = w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots \tag{3}$$

the system of second-order equations for w_0 , w_1, $\cdots$ is obtained, after introduction of eq. (3) in eq. (1), and not all four boundary conditions can be satisfied. This is a typical problem encountered in so-called singular perturbations.

It is clear that the difficulties originate at the edges where the first term in eq. (1) is of importance. The solution (3) is otherwise valid far away from $x = \pm 1$. Therefore, away from the edges the so-called global solution is

$$w_G = w_0 = \frac{1}{2} (1 - x^2) + O(\varepsilon)$$

In order to obtain the solution which is valid at the edges, we follow the general boundary-layer procedure [4] of shifting the origin to the region of nonuniformity and then magnifying the scale by introducing a new coordinate X instead of x , as follows.

Due to symmetry we consider the left-hand edge only (at $x = -1$) and introduce the new coordinate

$$X = \frac{x + 1}{\varepsilon^{\frac{1}{2}}} \ , \tag{5}$$

where the magnification factor $1/\varepsilon^{\frac{1}{2}}$ was chosen in such a way that in the new governing equation, given below, stiffness at the edge balances tension, rather than pressure [4]. Therefore, after introducing eq. (5) into eq. (1), the governing equation, valid near $x = -1$, is

$$\frac{d^4w}{dX^4} - \frac{d^2w}{dX^2} = \varepsilon \tag{6}$$

The solution of this equation is of the form

$$w_L = Ae^X + Be^{-X} + CX + D \tag{7}$$

After the exponentially growing term in eq. (7) is rejected (on physical grounds) and the boundary conditions (at $x = -1$) satisfied, the locally valid solution is

$$w_L = C(X - 1 + e^{-X}) + O(\varepsilon) \tag{8}$$

The constant C is unknown and is to be found next. This is, then, a typical situation in which a so-called matched-asymptotic method is used. But we now want to show that, in order to find C, the calculus of variation may be used to good advantage also.

The idea is as follows. The local solution w_L is valid in the region $-1 \le x \le x_A$, where x_A is a certain, still unknown, point close to the edge: $x_A = -1 + O(\varepsilon^{\frac{1}{2}})$, Fig. 1. The global solution w_G is valid in the region $x_A \le x \le 0$. (We consider only the left-hand side of the membrane.) The local solution w_L satisfies the boundary conditions at $x = -1$ exactly. Therefore, if one forms the variational statement of the problem and uses as trial solution w_L on $-1 \le x \le x_A$ and w_G on $x_A \le x \le 0$ all the boundary conditions will be satisfied and the so-called boundary terms in the variational statement will vanish. In that case the necessary calculations reduce to a simple integration as described below.

It is well known that the calculus of variations provides as good

results as the trial functions is a good approximation to the exact solution. Since w_G and w_L may be found to any order of accuracy, one is to expect that the calculus of variation should be able to provide the correct result for the constant C .

The variational statement for the system (1) and (2) is [5]:

$$\delta\left(\frac{U}{2}\right) = \delta \int_{x=-1}^{0} \left[\frac{\varepsilon}{2}\left(\frac{d^2w}{dx^2}\right)^2 + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 - w\right] dx + \text{B.T.} = 0 \tag{9}$$

Here, U is the potential energy of the membrane, δ is the variational symbol and B.T. stands for "boundary terms". We can write eq. (9) as

$$\delta\left(\frac{U}{2}\right) = \delta \int_{x=-1}^{0} \mathfrak{L}(w)dx + \text{B.T.} = 0 \tag{10}$$

where the operator $\mathfrak{L}$ is given by

$$\mathfrak{L} \equiv \frac{\varepsilon}{2}\left(\frac{d^2}{dx^2}\right)^2 + \frac{1}{2}\left(\frac{d}{dx}\right)^2 - 1 \tag{11}$$

The last integral, eq. (10), may be split up into two parts in terms of local and global solutions as

$$\delta\left(\frac{U}{2}\right) = \delta \int_{x=-1}^{x_A} \mathfrak{L}(w_L)dx + \int_{x=x_A}^{0} \mathfrak{L}(w_G)dx = 0 \tag{12}$$

Note that the boundary terms (B.T.) vanish, since w_L is used as the trial solution at $x = -1$. Note also that there are now two unknowns in the problem; x_A , the place of matching w_L and w_G and the constant C which is hidden in w_L in eq. (12).

w_L is now expressed in terms of x and the necessary integration in eq. (12) is carried out. Then the variation of the potential energy U with respect to two unknowns (x_A and C) is performed. The following system of equations is thus obtained:

$$- \frac{C}{\varepsilon^{\frac{1}{2}}} \left\{ \exp\left[- \frac{2}{\varepsilon^{\frac{1}{2}}} \left(x_A + 1 \right) \right] - 1 \right\} + \varepsilon^{\frac{1}{2}} \left(\frac{2C}{\varepsilon} + 1 \right) \left[\exp\left(- \frac{x_A + 1}{\varepsilon^{\frac{1}{2}}} - 1 \right) \right]$$

$$+ \left(\frac{C}{\varepsilon} + 1 \right) \left(x_A + 1 \right) - \frac{1}{\varepsilon^{\frac{1}{2}}} \left(x_A + 1 \right) - \frac{1}{2\varepsilon^{\frac{1}{2}}} \left(x_A^2 - 1 \right) = 0$$

$$\frac{C^2}{\varepsilon} \left\{ \exp\left[- \frac{2}{\varepsilon^{\frac{1}{2}}} \left(x_A + 1 \right) \right] \right\} - \left(\frac{C^2}{\varepsilon} + C \right) \exp\left(- \frac{x_A + 1}{\varepsilon^{\frac{1}{2}}} \right) + \frac{C^2}{2\varepsilon} + C - \frac{C}{\varepsilon^{\frac{1}{2}}}$$

$$- \frac{C}{\varepsilon^{\frac{1}{2}}} x_A - \frac{\varepsilon}{2} - x_A^2 + \frac{1}{2} = 0 \tag{13}$$

To obtain the solution for x_A and C we propose the following perturbation scheme:

$$C = a_0 + a_1 \varepsilon^{\frac{1}{2}} + 0(\varepsilon)$$

$$x_A = -1 + b_1 \varepsilon^{\frac{1}{2}} + 0(\varepsilon) \tag{14}$$

After substituting eq. (14) into eq. (13), equating the corresponding powers in ε, solving for a_o ; a_1 ; b_1 we get: $a_0 = 0$ and $a_1 = 1$ such that

$C = \varepsilon^{\frac{1}{2}}$. The same result for C is given in [4].

Note that C and x_A can not be more accurate than to order of $\varepsilon^{\frac{1}{2}}$ because we started calculations with w_G and w_L which are accurate up to order smaller than ε .

After w_G and w_L are known, the composite solution valid everywhere may be found by using Erdely's scheme, described in, say [1]. It is given in [4] as:

$$w = \frac{1}{2}(1 - x^2) - \varepsilon^{\frac{1}{2}}\left[1 - \exp\left(-\frac{1+x}{\varepsilon^{\frac{1}{2}}}\right) - \exp\left(-\frac{1-x}{\varepsilon^{\frac{1}{2}}}\right)\right]$$

2. TORSION OF SHAFT WITH KEYWAY

As a second example of the application of the calculus of variations in singular perturbations we consider a problem of determining the stress concentration factor for a circular cylindrical shaft in torsion with a semi-circular keyway cut along its length, Fig. 2. The exact solution for this problem may be found in [5] and [6], while the same problem is treated in [4] by using matched asymptotic expansions.

Mathematically, the problem reduces to that of finding the stress function ϕ such that it satisfies the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial t^2} = -1 \qquad (15)$$

over the cross-section of the shaft, and the boundary condition $\phi = 0$ along the boundaries of the cross-section, Fig. 2. Then the maximum shear stress $\tau = \frac{\partial\phi}{\partial y}$ is found at the root of the keyway and compared to the corresponding maximum stress of the circular cross-section (without the keyway).

We first find the global solution ϕ_G for a circular cross-section (without the keyway). It obeys the differential equation (15), subject to the boundary condition $\phi_G(r = 1) = 0$. The solution of this problem is easily found to be

$$\phi_G = \frac{1}{4}(1 - x^2 - y^2) \tag{16}$$

In order to find the local solution, valid in the immediate vicinity of the keyway (Fig. 3), we follow the same idea as in the previous example. First, the magnified coordinates are introduced as, [4]:

$$X = \frac{x}{\varepsilon} \quad , \quad Y = \frac{y + 1}{\varepsilon} \tag{17}$$

After introducing eq. (17) into eq. (15) the governing equation becomes

$$\frac{\partial^2 \phi_L}{\partial X^2} + \frac{\partial^2 \phi_L}{\partial Y^2} = -\varepsilon^2 \tag{18}$$

subject to the boundary condition $\phi_L = 0$ along the boundary shown in Fig. 3. This local solution is

$$\phi_L = C\left(Y - \frac{Y}{X^2 + Y^2}\right) + 0(\varepsilon^2) \tag{19}$$

The problem reduces now to that of finding the unknown constant C in eq. (19). In order to find C we again make use of the Rayleigh-Ritz procedure as in the previous problem.

Before performing the matching, we introduce polar global (r,θ)

and local (R,ψ) coordinates (Fig. 2 and Fig. 3):

$$x = r \sin\theta \ , \ y = -r\cos\theta \ , \ r^2 = x^2 + y^2 \ , \ \theta = \text{arc tang}\left(-\frac{x}{y}\right)$$

$$X = R \sin\psi \ , \ Y = R\cos\psi \ , \ R^2 = X^2 + Y^2 \ , \ \psi = \text{arc tg}\,\frac{X}{Y}$$

We then formulate the variational statement for the complete problem (with the keyway), and it is given in [5] as

$$\delta U = \delta \iint_D \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 - 2\phi \right] dx\,dy + \text{B.T.} \equiv \delta \iint_D \mathcal{L}(\phi)dx\,dy$$

$$+ \text{B.T.} = 0 \tag{20}$$

where D is the area of interest and B.T. stands, as before, for "boundary terms".

To assure that ϕ satisfies the boundary conditions everywhere, we write the variational statement in the form

$$\delta U = \delta \iint_{D_G} \mathcal{L}(\phi_G)dx\,dy + \delta \iint_{D_L} \mathcal{L}_1(\phi_L)dx\,dy \equiv \delta U_G + \delta U_L \tag{21}$$

where $\mathcal{L}(\phi)$ is defined by eq. (20) and

$$\mathcal{L}_1(\phi_L) = \left(\frac{\partial\phi}{\partial X}\right)^2 + \left(\frac{\partial\phi}{\partial Y}\right)^2 + 0(\varepsilon^2)$$

ϕ_G therefore satisfies the global boundary condition along the circle $(r = 1)$ while ϕ_L satisfies the local boundary condition along D_L (Fig. 3).

We now switch to the polar coordinate introduced previously, such that

$$\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 = \left(\frac{\partial}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial}{\partial \phi}\right)^2$$

$$\left(\frac{\partial}{\partial X}\right)^2 + \left(\frac{\partial}{\partial Y}\right)^2 = \left(\frac{\partial}{\partial R}\right)^2 + \frac{1}{R^2}\left(\frac{\partial}{\partial \psi}\right)^2$$

Also we will slightly modify regions D_G and D_L on which the integration in eq. (21) will actually be performed. Due to the symmetry of the problem, we are to integrate only across one-half of the cross-section. We assume that the global solution is valid inside the region CDEGC (Fig. 4), while the local solution is valid inside the region ABED . The two regions are divided by the circle of radius k (of order ε and still unknown) whose equation is

$$r = \cos\theta - (k^2 - \sin^2\theta)^{1/2}$$

and for small θ is given by

$$r = 1 - (k^2 - \theta^2)^{1/2}$$

When determining U_G we will first perform the integration across the semicircle CFG and subtract the corresponding integral along DFE . But the first integral will be a constant, say C_1 , and will not depend on the unknowns (k and C). Therefore, the variation of that integral

in U_G will be equal to zero. Thus, the integration necessary to find U_G will only be performed across the region FEDF , where $0 \leq \theta \leq k \cong tg\theta$ and r changes from $r = 1 - (k^2 - \theta^2)^{\frac{1}{2}}$ to $r = 1$.

Therefore, we find that

$$U_G = - \iint_{ABED} \left[\left(\frac{\partial \phi_G}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial \phi_G}{\partial \theta} \right)^2 - 2\phi_G \right] J_G \, dx \, dy + C_1 \tag{23}$$

where the Jacobian J_G is given by

$$J_G = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r \tag{24}$$

After the integration with respect to r is performed in eq. (23) and some higher order terms disregarded, the following result is obtained

$$\frac{1}{2} U_G = - \frac{1}{2} \int_0^k (k^2 - \theta^2)^{\frac{1}{2}} \, d\theta = - \frac{\pi}{8} k^2 + C_1 \tag{25}$$

The same procedure is now repeated with the local solution valid inside the region ABED where $0 \leq \psi \leq \frac{\pi}{2}$ and $1 \leq R \leq \frac{k}{\varepsilon}$. Therefore, one has to find U_L from

$$\frac{1}{2} U_L = \iint_{ABED} \left[\left(\frac{\partial \phi_L}{\partial R} \right)^2 + \frac{1}{R^2} \left(\frac{\partial \phi_L}{\partial \theta} \right)^2 \quad J_L \, dR \, d\psi \tag{26}$$

where the Jacobian $J_L = -R$ may be found, by using local coordinates and replacing them in eq. (24) instead of global ones. ϕ_1 is given by eq. (19) and in polar coordinates as:

$$\phi_L = C(R - \frac{1}{R}) \cos \psi \tag{27}$$

After the necessary integration in (26) is completed, one finds that

$$\frac{1}{2} U_L = \frac{\pi C^2}{4}\left(\frac{k^2}{\varepsilon^2} - \frac{\varepsilon^2}{k^2}\right) - \pi C \varepsilon^2\left(\frac{k^3}{3\varepsilon^3} - \frac{k}{\varepsilon} + \frac{2}{3}\right) \tag{28}$$

Knowing U_G and U_L from eq. (25) and (28), the total potential energy may be found as

$$U(k,C) = U_L + U_G \tag{29}$$

Performing the variation of U with respect to two unknowns (k and C), one obtains the following equations

$$\frac{C^2}{4}\left(2\,\frac{k}{\varepsilon^2} + 2\,\frac{\varepsilon^2}{k^3}\right) - C\varepsilon\left(\frac{k^2}{\varepsilon^2} - 1\right) + \frac{1}{4}\,k = 0$$

$$C\left(\frac{k^2}{\varepsilon^2} - \frac{\varepsilon^2}{k^2}\right) - 2\varepsilon^2\left(\frac{k^3}{3\varepsilon^2} - \frac{k}{\varepsilon} + \frac{2}{3}\right) = 0$$

One can again use the perturbation scheme such as in eq. (14) to find that $k = \varepsilon + O(\varepsilon^2)$ and $C = \frac{1}{2}\,\varepsilon + O(\varepsilon^2)$. The same result for C was obtained in [4] and that is in essence what we wanted to show.

The composite expansion is found again by using Erdely's scheme [1] and is given by

$$\phi = \frac{1}{4}\,(1 - x^2 - y^2) - \frac{1}{2}\,\varepsilon^2\,\frac{y + 1}{x^2 + (1 + y)^2}$$

The maximum shearing stress (at the bottom of the keyway) is given as

$$\tau_K = \frac{\partial\phi}{\partial y} = 1 + 0(\varepsilon)$$

while in the case when there is no keyway the corresponding stress is $\tau_{wk} = \frac{\partial\phi_G}{\partial y} = \frac{1}{2}$. Therefore, the stress concentration factor is $\tau_k/\tau_{wk} = 2$.

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FIGURE CAPTIONS

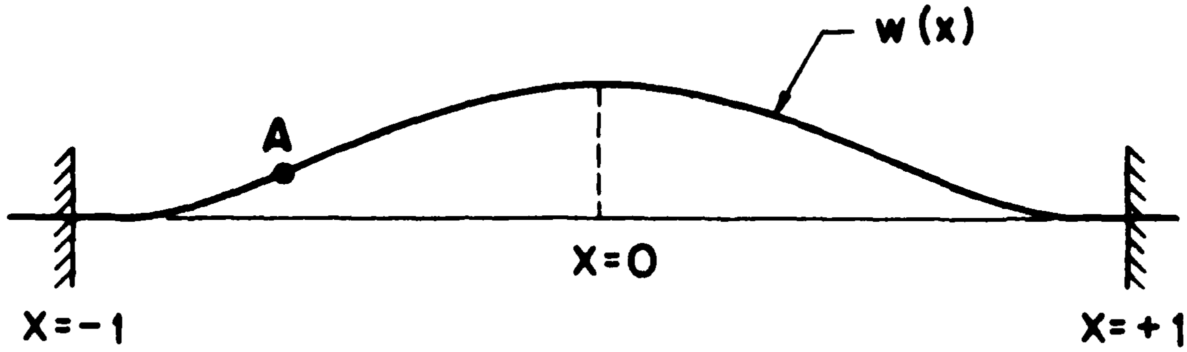
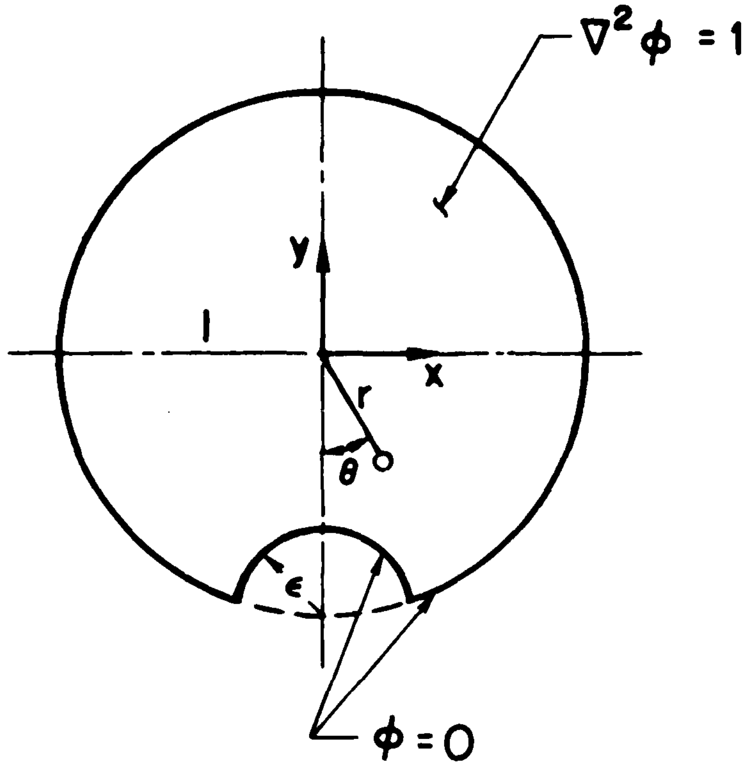


FIGURE 1



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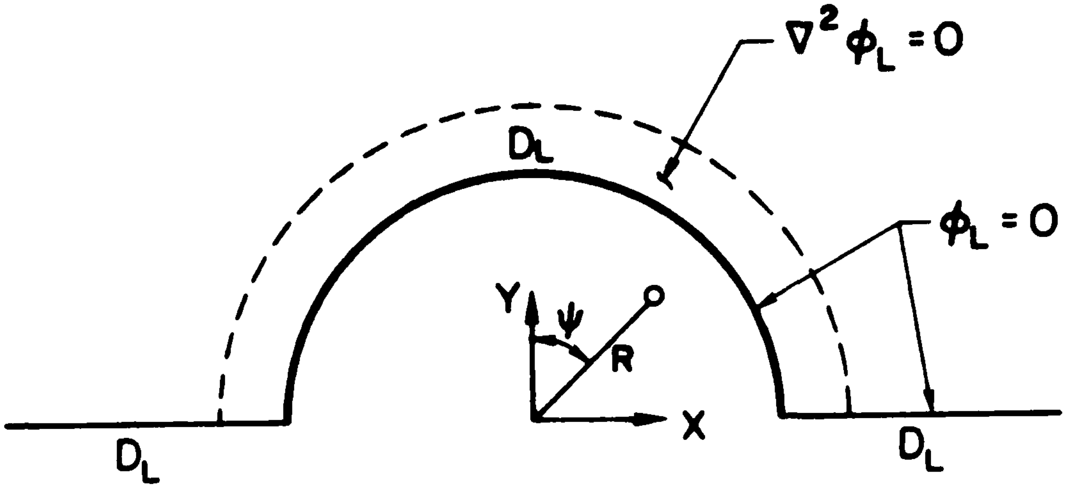

FIGURE 3

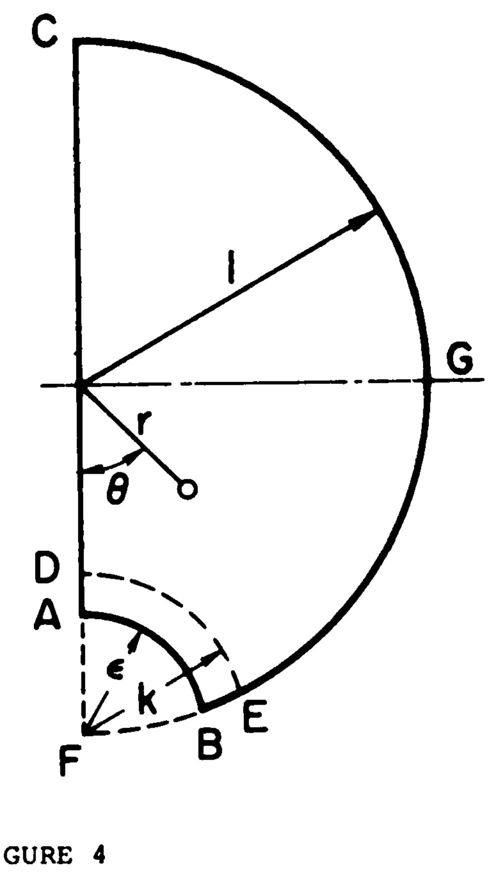

FIGURE 4